Proposition: The difference between a natural number k and the sum of its digits written in base b is divisible by $(b-1)$.

Proof: Let k be written as the sum of powers of the base b:

$$k = c_0 b^0 + c_1 b^1 + \ldots + c_n b^n = \sum_{i=0}^{n} c_i b^i$$

The sum of the digits of k written in base b is

$$s = c_0 + c_1 + \ldots + c_n = \sum_{i=0}^{n} c_i$$

Then

$$\begin{aligned} k - s &= \sum_{i=0}^{n} c_i b^i - c_i \\ &= \sum_{i=0}^{n} c_i (b^i - 1) \end{aligned}$$

By the lemma below, $b^i = b^0 + (b-1)\sum_{j=0}^{i-1} b^j$, and therefore

$$\begin{aligned} k - s &= \sum_{i=0}^{n} c_i \left[b^0 + \left((b-1) \sum_{j=0}^{i-1} b^j \right) - 1 \right] \\ &= (b-1) \sum_{i=0}^{n} \sum_{j=0}^{i-1} c_i b^j \end{aligned}$$

which is divisible by $(b-1)$.

Lemma: $b^n = b^0 + (b-1)\sum_{i=0}^{n-1} b^i$

Note that

$$b^n = b \cdot b^{n-1} = (b-1+1)b^{n-1} = (b-1)b^{n-1} + b^{n-1}$$

This applies recursively to b^{n-1}, b^{n-2}, and so on, until $b^1 = (b-1)b^0 + b^0$. Substituting into the original equation,

$$\begin{aligned} b^n &= (b-1)b^{n-1} + (b-1)b^{n-2} + \ldots + (b-1)b^1 + (b-1)b^0 + b^0 \\ &= b^0 + (b-1) \sum_{i=0}^{n-1} b^i \end{aligned}$$

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